

Single scalar cosmology

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April 18, 2013

Abstract

The cosmology of flat FLRW universes dominated by a single scalar field is discussed. General features of the evolution of the universe and the scalar field are illustrated by specific examples. In particular the role of critical points, where the scalar field is stationary, is emphasized and their classification and interpretation as stable or unstable end points, or turning points is explained.

1 Scalar fields in cosmology

The discovery of the Higgs particle [1, 2] has confirmed that scalar fields play a fundamental role in subatomic physics. Therefore they must also have been present in the early universe and played a part in its development [3, 4, 5, 6, 7]. Significant evidence for a period of inflation is provided by observations of the CMB [8, 9, 10]. About scalar fields on present cosmological scales nothing is known, but in view of the observational evidence for accelerated expansion [11, 12] it is quite well possible that they take part in shaping our universe now and in the future [13, 14].

Understanding the impact of scalar fields on the evolution of the cosmos is therefore of direct observational relevance. There is already a vast literature on the subject, e.g. [15]-[23] and references therein. In this paper I confine myself to a summary of some recent work concerning the simplest scenario, the evolution of a flat, isotropic and homogeneous universe in the presence of a single cosmic scalar field. Neglecting ordinary matter and radiation, the evolution of such a universe is described by two degrees of freedom, the homogeneous scalar field $\varphi(t)$ and the scale factor of the universe $a(t)$. The relevant evolution equations are the Friedmann and Klein-Gordon equations, reading¹

$$\frac{1}{2}\dot{\varphi}^2 + V = 2H^2, \quad \ddot{\varphi} + 3H\dot{\varphi} + V' = 0, \quad (1)$$

where $V[\varphi]$ is the potential of the scalar fields, and $H = \dot{a}/a$ is the Hubble parameter. Furthermore, an overdot denotes a derivative w.r.t. time, whilst a prime denotes a derivative w.r.t. the scalar field φ .

For single-field models in which the scalar $\varphi(t)$ is a single-valued function in some interval of time, it is possible to reparametrize the Hubble parameter in terms of φ :

$$H(t) = H[\varphi(t)]. \quad (2)$$

This will be assumed in the following. Now taking time derivatives, we arrive at the results

$$\dot{\varphi}(\ddot{\varphi} + V') = 6H\dot{H}, \quad \dot{H} = H'\dot{\varphi}. \quad (3)$$

It follows, that for $\dot{\varphi} \neq 0$ and $H \neq 0$ one gets

$$\dot{\varphi} = -2H', \quad \dot{H} = -\frac{1}{2}\dot{\varphi}^2 \leq 0. \quad (4)$$

Thus the Hubble parameter is a semi-monotonically decreasing function of time. Finally, replacing the time derivatives in the Friedmann equation we find

$$V = 3H^2 - 2H'^2. \quad (5)$$

Given $V[\varphi]$ this is a first-order differential equation for $H[\varphi]$. In the following we discuss the solutions of the is equation.

¹We use units in which $c = \hbar = 8\pi G = 1$.

2 Stationary points

As we have observed, $H(t)$ is a decreasing function of time, except at stationary points where $\dot{\varphi} = -2H' = 0$. It follows first of all, that as long as H is positive, the expansion of the universe will generically slow down, whilst in case $H(t)$ crosses into the domain of negative values collapse of the universe becomes inevitable. Therefore the existence or non-existence of stationary points is relevant to the question of the ultimate fate of our simple model universes.

There are two kinds of stationary points; a point where $\dot{\varphi} = H' = 0$ is an end point of the evolution if

$$\ddot{\varphi} = 4H'H'' = 0, \quad (6)$$

which happens if H'' is finite. In contrast, if

$$\ddot{\varphi} = 4H'H'' \neq 0, \quad (7)$$

H'' necessarily diverges in such a way as to make $\ddot{\varphi}$ finite: $H'' \propto 1/H'$. This property is illustrated by the remarkable example of the eternally oscillating scalar field:

$$\varphi(t) = \varphi_0 \cos \omega t. \quad (8)$$

For such a scalar field to exist we require

$$H' = -\frac{1}{2} \dot{\varphi} = \frac{\omega \varphi_0}{2} \sin \omega t = \frac{\omega}{2} \sqrt{\varphi_0^2 - \varphi^2}. \quad (9)$$

It is clear, that there are infinitely many stationary points

$$\omega t_n = n\pi, \quad \varphi(t_n) = (-1)^n \varphi_0, \quad (10)$$

where $H' = 0$. Now

$$H'' = -\frac{1}{2} \frac{\omega \dot{\varphi}}{\sqrt{\varphi_0^2 - \varphi^2}}, \quad (11)$$

and therefore H'' diverges at all stationary points t_n , but in such a way that

$$4H'H'' = -\omega^2 \varphi = \ddot{\varphi}. \quad (12)$$

We conclude that all stationary points (10) are turning points.

However, though the scalar field in this example oscillates forever with constant frequency and amplitude, this does not imply that the scale factor oscillates as well; on the contrary, the Hubble parameter ultimately moves into the domain of negative values and the universe collapses:

$$\begin{aligned} H &= H_0 - \frac{1}{4} \omega \varphi_0^2 \arccos \left(\frac{\varphi}{\varphi_0} \right) + \frac{1}{4} \omega \varphi \sqrt{\varphi_0^2 - \varphi^2} \\ &= H_0 - \frac{1}{4} \omega^2 \varphi_0^2 t + \frac{1}{8} \omega \varphi_0^2 \sin 2\omega t. \end{aligned} \quad (13)$$

The corresponding solution for the scale factor is

$$a(t) = a(0)e^{H_0 t - \frac{1}{8}\omega^2\varphi_0^2 t^2 + \frac{1}{16}(1 - \cos 2\omega t)}, \quad (14)$$

which is a gaussian, slightly modulated by an oscillating function of time; see fig. 1.

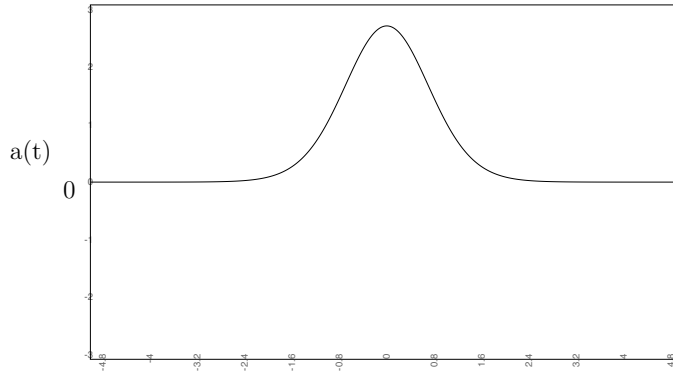


Fig. 1: Scalefactor $a(t)$ for an eternally oscillating scalar field.

The potential giving rise to this behaviour can also be constructed:

$$\begin{aligned} V &= 3H^2 - 2H'^2 \\ &= 3 \left(H_0 - \frac{1}{4}\omega\varphi_0^2 \arccos\left(\frac{\varphi}{\varphi_0}\right) + \frac{1}{4}\omega\varphi\sqrt{\varphi_0^2 - \varphi^2} \right)^2 - \frac{\omega^2}{2}(\varphi_0^2 - \varphi^2). \end{aligned} \quad (15)$$

Observe, that this potential keeps track of the number of oscillations the scalar field has performed through the arccos-function, so ultimately V increases indefinitely as a function of time, whilst the volume of a representative domain of space decreases rapidly.

3 The fate of the universe

The arguments presented above bring out that there are only a few final states for universes governed by a single scalar field at large times. Once H becomes negative the collapse of the universe becomes inevitable; if H never becomes negative, it must tend to a vanishing or positive final minimum, which can be reached either in finite or infinite time. The universe then ends up in a Minkowski or in a de Sitter state. These conclusions are a consequence of the non-positivity of \dot{H} , eq. (4), which implies that a negative H can never return to larger values at later times [22, 23].

In order to establish the existence of end points or asymptotic end points of evolution at non-negative values of H , we first consider the locus of all stationary points, defined by

$$\dot{\varphi} = -2H' = 0 \quad \Rightarrow \quad V = 3H^2 \geq 0. \quad (16)$$

It follows that stationary points can occur only in the region of positive or vanishing potential. In particular this holds for end points, which therefore do not occur in a region of negative potential. Moreover, it is clear that a Minkowski final state occurs only at a stationary point where $V = 0$, whereas all stationary points with $V > 0$ correspond to de Sitter states. To correspond to an end point of the evolution, H'' must be finite at these stationary points to guarantee that $\ddot{\varphi} = 0$ as well.

Next observe, that eq. (5) implies

$$V' = 2H'(3H - 2H''), \quad (17)$$

and therefore $V' = 0$ if $H' = 0$ and H and H'' are finite. As a result end points of the evolution necessarily occur at an extremum of V , but only if $V \geq 0$ there.

We can illustrate these results by the example of quadratic potentials

$$V = v_0 + \frac{m^2}{2} \varphi^2. \quad (18)$$

We distinguish the cases $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$. The stationary points (16) are represented graphically by the curves in the φ - H -plane in fig. 2.

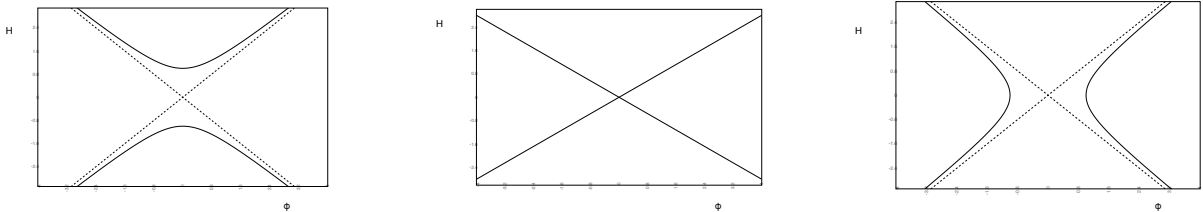


Fig. 2: Critical curves $H'[\varphi] = 0$ for quadratic potentials (18) with $v_0 > 0$ (left), $v_0 = 0$ (middle) and $v_0 < 0$ (right).

First, for $v_0 > 0$ there exists a stationary point for any value of φ , but the potential has a unique minimum at $\varphi = 0$, which is the only stationary point where $V' = 0$, and therefore the only end point. Indeed, once this point is reached H can not decrease anymore and we have final state of de Sitter type. For $v_0 = 0$ the critical curves become straight lines, crossing at the origin where $H = 0$ at $V = 0$. This is still a stationary point with $\ddot{\varphi} = 0$ representing a Minkowski state, but as V' is not defined there it is really to be interpreted as a limit of the previous case. There are no evolution curves flowing from the domain $H > 0$ to the domain $H < 0$. Finally, for $v_0 < 0$ there are no stationary points in the region $\varphi^2 < 2|v_0|/m^2$, and the solutions can cross into the domain of negative H there.

Fig. 3 shows some actual solutions for the cases of $v_0 > 0$ and $v_0 < 0$, respectively. These solutions can be obtained by making a power series expansion [23]

$$H[\varphi] = h_0 + h_1\varphi + h_2\varphi^2 + h_3\varphi^3 + \dots \quad (19)$$

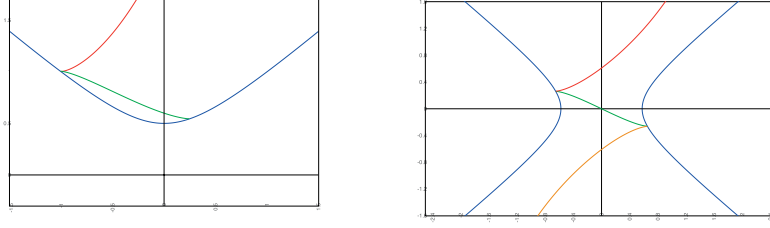


Fig. 3: Solutions $H[\varphi]$ for quadratic potentials (18) with $v_0 > 0$ (left), and $v_0 < 0$ (right).

Substitution into eq. (5) then leads to the equalities

$$3h_0^2 - 2h_1^2 = v_0, \quad h_1(3h_0 - 4h_2) = 0, \quad 4h_1(h_1 - 4h_3) + \frac{8}{3}h_2(3h_0 - 4h_2) = m^2, \quad \dots, \quad (20)$$

from which the solutions $H[\varphi]$ can be reconstructed. The same information can be used to calculate the total expansion factor of the universe, as defined by the number of e -folds in some interval of time:

$$N = \int_1^2 dt H = - \int_1^2 d\varphi \frac{H}{2H'} = -\frac{1}{2} \int_1^2 d\varphi \frac{h_0 + h_1\varphi + h_2\varphi^2 + \dots}{h_1 + 2h_2\varphi + 3h_3\varphi^2 + \dots}. \quad (21)$$

This number can get sizeable contributions only in regions where the slow-roll condition is satisfied:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{2H'^2}{H^2} < 1 \quad \Rightarrow \quad 3H^2 - V < H^2. \quad (22)$$

Thus we simultaneously have

$$V < 3H^2 \quad \text{and} \quad V > 2H^2 \quad \Leftrightarrow \quad 0 \leq \frac{V}{3} < H^2 < \frac{V}{2}. \quad (23)$$

In most cases this holds only for a relatively narrow range of field values.

4 Unstable endpoints and inflation

In our discussion we have carefully distinguished between end points and turning points of the scalar field evolution. In both cases $\dot{\varphi} = 0$, but at end points in addition $\ddot{\varphi} = 0$, which can happen only at extrema of the potential $V[\varphi]$. However, if the end point occurs at a relative maximum or saddle point of the potential, this end point will be classically unstable. Indeed, the field can remain there for an indefinite period of time, but any slight change in the initial conditions will cause it to move on to lower values of the Hubble parameter. Nevertheless, such a period of temporary slow roll of the field creates the right conditions for a period of inflation. We finish this paper by discussing an example of this kind taken from ref. [17].

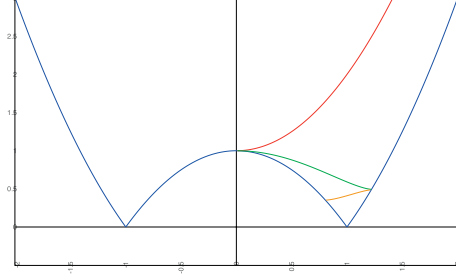


Fig. 4: Critical curves of stationary points and solutions $H[\varphi]$ for a quartic potential with spontaneous symmetry breaking.

This example is tailored to provide an exponentially decaying scalar field

$$\varphi(t) = \varphi_0 e^{-\omega t}. \quad (24)$$

The Hubble parameter and potential giving rise to this solution can be constructed following the same procedure as for the eternally oscillating field in sect. 2, with the result

$$H = h + \frac{1}{4} \omega \varphi^2, \quad V[\varphi] = v_0 - \frac{\mu^2}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4, \quad (25)$$

where

$$v_0 = 3h^2, \quad \mu^2 = \omega^2 - 3\omega h, \quad \lambda = \frac{3\omega^2}{4}. \quad (26)$$

Thus we obtain a quartic potential; for $\mu^2 > 0$ it has minima in which reflection symmetry is spontaneously broken. The exponential solution (24) ends asymptotically at the unstable maximum of the potential where $\dot{\varphi} = \ddot{\varphi} = 0$. As such it represents an end point of the evolution, but a minimal change in the initial conditions for the scalar field will turn the end point into a reflection point (if it starts at lower H), or it will overshoot the maximum (if it starts at higher H). Thus the end point is unstable, but the exponential decay (24) will still provide a good approximation to first part of the evolution of the universe for solutions $H[\varphi]$ coming close to the maximum of the potential.

Next observe, that the Hubble parameter (25) in combination with the exponential scalar field (24) leads to a behaviour of the scale factor given by

$$a(t) = a_0 e^{ht + \frac{1}{8} \varphi_0^2 (1 - e^{-2\omega t})}. \quad (27)$$

Thus for $h > 0$ this epoch in the evolution of the universe ends in an asymptotic de Sitter state with Hubble constant h . Afterwards, the scalar field will roll further down the potential; provided $3h \leq \omega \leq 6h$ it will oscillate around the minimum until it comes to rest in another de Sitter or a Minkowski state, again depending on the value of h . In particular, for $\omega \geq 3h$ the model has a final de Sitter or Minkowski state in which $\dot{\varphi} = 0$ and

$$\langle \varphi^2 \rangle = \frac{\mu^2}{\lambda} = \frac{4}{3} \left(1 - \frac{3h}{\omega} \right). \quad (28)$$

In this final state the energy density is

$$\langle V \rangle = v_0 - \frac{\mu^4}{4\lambda} = \frac{\omega}{3} (6h - \omega). \quad (29)$$

On the other hand, the energy density for the solution (24) is

$$\rho_s(t) = \frac{1}{2} \dot{\varphi}^2 + V = 3H^2 = 3 \left(h + \frac{1}{4} \omega \varphi_0^2 e^{-2\omega t} \right)^2. \quad (30)$$

Now the solution (27) for the scale factor shows, that before reaching the first turning point at $\varphi = 0$ the scale factor increases by an additional number of e -folds given by

$$N = \frac{1}{8} \varphi_0^2. \quad (31)$$

Therefore the initial energy density at $t = 0$ can be written as

$$\rho_s(0) = 3(h + 2N\omega)^2. \quad (32)$$

If we take this initial energy density to equal the Planck density: $\rho_s(0) = 1$, this establishes a simple relation between h , ω and N .

Another relation is obtained by taking $\langle V \rangle$ equal to the observed energy density of the universe today [10]:

$$\langle V \rangle = 3H_0^2 = 1.04 \times 10^{-120} \quad (33)$$

in Planck units. Being so close to zero it implies that to an extremely good approximation $\omega = 6h$, and

$$3h^2(1 + 12N)^2 = 1, \quad \mu^2 = 18h^2, \quad \lambda = 27h^2. \quad (34)$$

The lower limit on N for inflation as derived from the CMB observations is $N \geq 60$, which requires

$$h \leq 0.8 \times 10^{-3}. \quad (35)$$

Now expanding φ around its vacuum expectation value

$$\varphi = \frac{\mu}{\sqrt{\lambda}} + \chi, \quad (36)$$

the potential becomes

$$V = \frac{1}{2} m_\chi^2 \chi^2 + \frac{\alpha}{3} m_\chi \chi^3 + \frac{\lambda}{4} \chi^4, \quad (37)$$

where

$$m_\chi = 6h, \quad \alpha = 9h\sqrt{\frac{3}{3}}, \quad \lambda = 27h^2. \quad (38)$$

According to (35) the upper limits on these parameters are

$$m_\chi = 0.48 \times 10^{-2}, \quad \alpha = 0.88 \times 10^{-2}, \quad \lambda = 0.17 \times 10^{-4}. \quad (39)$$

Converting to particle physics units, the upper limit on the mass is $m_\chi \leq 1.2 \times 10^{-16}$ GeV. This suggests that the inflaton could be associated with a GUT scalar of Brout-Englert-Higgs type [17].

Acknowledgement

The work described in this paper was presented at the 25th meeting on Physics Beyond the Standard Model (Bad Honnef, Germany; March 20, 2013). It is part of the research program of the Foundation for Fundamental Research of Matter (FOM).

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